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Orientably regular maps with given exponent group

Marston D. E. Conder and Jozef Širáň

ABSTRACT

We prove that for every integer $d \geq 3$ and every group U of units mod d , there exists an orientably regular map of valency d with exponent group U .

1. Introduction

An *orientably regular map* M is a 2-cell embedding of a connected graph in an orientable surface, such that the group of all orientation-preserving automorphisms $\text{Aut}^+ M$ of the embedding acts as regularly (sharply transitively) on the set of arcs of the graph. It follows that every vertex of M has the same valency, say d , and every face of M is bounded by a closed walk of the same length, say m .

If e is an arc at any vertex v of M , then regularity implies that $\text{Aut}^+ M$ contains an involution x acting like a 180-degree rotation of M about the centre of e , and an element y of order d acting like a d -fold rotation of M about v . Then by connectivity, the group $\text{Aut}^+ M$ is generated by x and y , and admits a presentation of the form $\text{Aut}^+ M = \langle x, y \mid x^2 = y^d = (xy)^m = \dots = 1 \rangle$. The pair (d, m) is called the *type* of the map. Conversely, given any generating pair (x, y) for a group G with the above form, one may construct an orientably regular map M with $\text{Aut}^+ M = G$ by taking edges, vertices and faces of M as the (right) cosets in G of the subgroups $\langle x \rangle$, $\langle y \rangle$ and $\langle xy \rangle$, respectively, and with incidence given by non-empty intersection of cosets. (Also the arcs may be taken as the elements of G .) Thus, orientably regular maps of valency d and face length m may be identified with 2-generator group presentations of the form $\langle x, y \mid x^2 = y^d = (xy)^m = \dots = 1 \rangle$.

Fundamentals of the theory of maps and orientably regular maps are explained in [8], some deep connections between such maps, Riemann surfaces and Galois groups are described in detail in [9], and a recent survey containing a large number of facts about regular maps is given in [11].

Next, let M and $G = \text{Aut}^+ M = \langle x, y \rangle$ be as above. An integer j relatively prime to d is said to be an *exponent* of M if the assignment $(x, y) \mapsto (x, y^j)$ extends to an automorphism of G . Algebraically, this means that (x, y) and (x, y^j) satisfy the relations as each other, while from the point of view of maps, it means that if a new map M^j is constructed from M by replacing the clockwise local cyclic order π_v of arcs at each vertex v by π_v^j , then resulting map M^j is isomorphic to M . Orientably regular maps admitting the exponent -1 are isomorphic to their mirror image, and are therefore called *reflexible*.

The collection of all exponents of M forms a subgroup of the group of units \mathbb{Z}_d^* , and is called the *exponent group* of M . The notion of an exponent was introduced in [10], with applications in the classification of orientably regular maps with a given underlying graph. Previously, the

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mapping $M \rightarrow M^j$ (even in the case when the two maps may not be isomorphic) was known as a *hole operator*, and studied by Wilson [14], but this mapping has also been attributed to Coxeter.

For the exponents of an orientably regular map of given valency d , there are two ‘extremes’: one where the exponent group is trivial, or consists only of 1 and -1 , and the other where the map admits the ‘full’ exponent group Z_d^* .

In [2], it was shown that for every $d \geq 3$ there are infinitely many finite orientably regular maps of valency d with trivial exponent group. This was done with the help of a method that allows one to forbid the creation of new automorphisms in lifted maps, but unfortunately the method offers no control over the face length. Also it was proved in [12] using residual finiteness of triangle groups that for every pair of positive integers d and m with $1/d + 1/m \leq 1/2$, there exist infinitely many finite orientably regular and reflexible maps of type (d, m) that admit no exponents other than 1 and -1 .

At the other end of the spectrum, it was shown in [13] that for every integer $d \geq 3$ there exist infinitely many finite orientably regular maps with exponent group Z_d^* . Again this was achieved using residual finiteness of triangle groups, but this time losing control over the face length of resulting maps. Such maps were called ‘kaleidoscopic’ in [1], where a covering construction was given for a kaleidoscopic d -valent regular map invariant also under duality and Petrie duality, for every even d . A different construction for such ‘super-symmetric’ d -valent maps was given for an infinite set of odd values of d in [6].

In this paper, we deal with the ‘intermediate’ cases, by considering arbitrary subgroups of the group of units modulo the valency d . We prove that for every $d \geq 3$ and every given subgroup U of Z_d^* , there exist infinitely many finite orientably regular maps of valency d with exponent group equal to (and not just isomorphic to) U .

2. The main result

THEOREM 1. *For every $d \geq 3$ and every subgroup U of Z_d^* , there are infinitely many finite orientably regular maps of degree d with exponent group equal to U .*

Proof. Let G be the free product $Z_2 * Z_d$ of the cyclic groups of order 2 and $d \geq 3$, with presentation $\langle X, Y \mid X^2 = Y^d = 1 \rangle$, and let $D = G'$ be the derived subgroup of G , of index $2d$ in G , with quotient $G/D \cong Z_2 \times Z_d$. By Reidemeister–Schreier theory [5], the group D is free of rank $d - 1$, generated by the commutators $W_j = [X, Y^j]$ for $j \in \{1, 2, \dots, d - 1\}$.

We will construct for any given subgroup U of Z_d^* an infinite family of quotients of G that give rise to orientably regular maps of degree d with exponent group U .

For any prime p , let $N_p = D'D^{(p)}$ be the subgroup of D generated by the commutators and p th powers of all elements of D . This subgroup is characteristic in D and hence normal in G , and the quotient D/N_p is isomorphic to the direct product Z_p^{d-1} of $d - 1$ copies of Z_p . Also G/N_p is an extension of $D/N_p \cong Z_p^{d-1}$ by $(G/N_p)/(D/N_p) \cong G/D \cong Z_2 \times Z_d$, and hence G/N_p has order $2dp^{d-1}$.

Next, for any $u \in Z_d^*$, let k_u be the automorphism of G that takes the generating pair (X, Y) to the generating pair (X, Y^u) . Note that this permutes the generators $W_j = [X, Y^j]$ of D among themselves, and therefore preserves D , and its characteristic subgroup N_p , and so induces an automorphism h_u of $G_p = G/N_p$, with $(Ng)^{h_u} = N(g^{k_u})$ for all $g \in G$.

Now, let U be any subgroup of Z_d^* . Then, $K_U = \{k_u : u \in U\}$ and $H_U = \{h_u : u \in U\}$ are groups of automorphisms of G and G_p (respectively), both isomorphic to U .

We will show that if the prime p is congruent to 1 mod d , then there exists a normal subgroup L_U of $G_p = G/N_p$ contained in D/N_p such that L_U is preserved by H_U , and furthermore, that L_U can be chosen so that it is not preserved by h_r for any $r \in Z_d^* \setminus U$. Under these

circumstances, the quotient G_p/L_U determines a finite orientably regular map M of valency d with exponent group containing U , and then finally, we will show that the exponent group of M is equal to U . We break this up into three steps below.

Step 1. Let x and y be the images of X and Y under the natural quotient homomorphism from G to $G/N_p = G_p$, and let $w_j = [x, y^j] = xy^{-j}xy^j$, which is the image of $W_j = [X, Y^j]$, for $j \in \{1, 2, \dots, d-1\}$. Then, these w_j are elements of the elementary abelian p -group $V_p = D/N_p \cong Z_p^{d-1}$, and so commute with each other. Moreover, it is easy to see that $xw_jx = y^{-j}xy^jx = w_j^{-1}$ and $y^{-1}w_jy = y^{-1}xy^{-j}xy^{j+1} = y^{-1}xyxy^{-(j+1)}xy^{j+1} = w_1^{-1}w_{j+1}$, for all $j \in \{1, 2, \dots, d-1\}$, if we define also $w_d = [x, y^d] = 1$.

Next, suppose $p \equiv 1 \pmod d$, and let t be any non-trivial d th root of 1 mod p , so that $1 + t + t^2 + \dots + t^{d-1} \equiv 0 \pmod p$. Define

$$v_t = w_1^t w_2^{t^2} \dots w_{d-2}^{t^{d-2}} w_{d-1}^{t^{d-1}},$$

which is an element of the abelian p -group $V_p = D/N$. Conjugation by x inverts v_t , while

$$\begin{aligned} y^{-1}v_t y &= (y^{-1}w_1 y)^t (y^{-1}w_2 y)^{t^2} \dots (y^{-1}w_{d-2} y)^{t^{d-2}} (y^{-1}w_{d-1} y)^{t^{d-1}} \\ &= (w_1^{-1}w_2)^t (w_1^{-1}w_3)^{t^2} \dots (w_1^{-1}w_{d-1})^{t^{d-2}} (w_1^{-1})^{t^{d-1}} \\ &= w_1^{-(t+t^2+\dots+t^{d-2}+t^{d-1})} w_2^t w_3^{t^2} \dots w_{d-2}^{t^{d-3}} w_{d-1}^{t^{d-2}} \\ &= w_1 w_2^t w_3^{t^2} \dots w_{d-2}^{t^{d-3}} w_{d-1}^{t^{d-2}} \\ &= (v_t)^{t^{-1}}. \end{aligned}$$

It follows that the cyclic subgroup L_t of $V_p = D/N_p$ generated by v_t is normal in G_p .

Now, take $L_U = \langle L_t^{h_u} : u \in U \rangle$. Since L_t is a normal subgroup of G_p , the image $L_t^{h_u}$ of L_t under each automorphism h_u is also a normal subgroup of G_p , and hence L_U is normal in G_p . Moreover, L_U is clearly preserved by H_U , as required.

Step 2. Suppose further that t is a *primitive* d th root of 1 mod p , and for each $j \in Z_d^*$, define the element $v_t^{(j)}$ of V_p by

$$v_t^{(j)} = h_{j-1}(v_t) = \prod_{i \in Z_d^*} h_{j-1} \left(w_i^{t^i} \right) = \prod_{i \in Z_d^*} (w_{j-1i})^{t^i} = \prod_{\ell \in Z_d^*} w_\ell^{(t^j)^\ell}.$$

We claim that these $\phi(d) = |Z_d^*|$ elements $v_t^{(j)}$ generate a subgroup of order $p^{\phi(d)}$ in V_p , or equivalently, that they are linearly independent over Z_p when V_p is considered as a vector space over Z_p of dimension $d-1$. To see this, if we take the set $\{w_1, w_2, \dots, w_{d-1}\}$ as a basis for V_p , and write any element $w_1^{a_1} w_2^{a_2} \dots w_{d-1}^{a_{d-1}}$ of V_p as a $(d-1)$ -tuple $(a_1, a_2, \dots, a_{d-1})$, then by its definition above, $v_t^{(j)}$ can be written as the $(d-1)$ -tuple $(t^j, t^{2j}, \dots, t^{(d-1)j})$. Hence, the set $\{v_t^{(j)} : j \in Z_d^*\}$ can be represented by a $\phi(d) \times (d-1)$ sub-matrix of the Vandermonde matrix

$$\begin{pmatrix} t & t^2 & t^3 & \dots & t^{d-1} \\ t^2 & t^4 & t^6 & \dots & t^{2(d-1)} \\ t^3 & t^6 & t^9 & \dots & t^{3(d-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t^{d-1} & t^{2(d-1)} & t^{3(d-1)} & \dots & t^{(d-1)(d-1)} \end{pmatrix}.$$

This matrix has determinant $\prod_{1 \leq i < j \leq d-1} (t^j - t^i)$, which is non-zero in Z_p since t is a primitive d th root of 1 mod p , and it follows that for any subset S of Z_d^* , the rows with first entry t^j with $j \in S$ are linearly independent over Z_p . In particular, taking $S = Z_d^*$, we see the above claim is true.

But also this shows that $h_r(L_U) \neq L_U$ for any $r \in \mathbb{Z}_d^* \setminus U$, because if $h_r(L_U) = L_U$, then $L_U = h_{r^{-1}}(L_U)$ and so the vector corresponding to $v_t^{(r)} = h_{r^{-1}}(v_t)$ is a linear combination of the vectors corresponding to the elements $v_t^{(u)}$ for $u \in U$, which is impossible.

Step 3. It remains to show that the exponent group of the orientable-regular map M arising from the quotient G_p/L_U of G is equal to U . By Step 1, we know that this exponent group contains U . To prove the reverse inclusion, suppose that j is any exponent of this map M . Then also j^{-1} is an exponent of M , and hence there exists an automorphism θ of G_p/L_U that fixes the element xL_U and takes yL_U to $y^{j^{-1}}L_U$. But now $v_t \in L_t \subseteq L_U$, so the coset v_tL_U is trivial in G_p/L_U , and it follows that the coset containing $v_t^{(j)} = h_{j^{-1}}(v_t)$ is trivial as well. Thus $v_t^{(j)}$ lies in L_U , and by Step 2, we deduce that $j \in U$.

This completes the proof.

3. Concluding remarks

The method we have used does not enable control over the face length of the resulting maps. This is no accident, as it is *not* true that there exist orientably regular maps of given type (d, m) with $1/d + 1/m \leq \frac{1}{2}$ and having a given exponent group. For example, in the case of triangulations (with $m = 3$), it was shown in [13] that an orientably regular map of type $(d, 3)$ with valency $d \equiv \pm 1 \pmod{6}$ cannot have more than $\phi(d)/2$ exponents, and that if d is a prime such that $d \equiv -1 \pmod{8}$ and $(d-1)/2$ is also prime, then such a triangulation cannot have exponents other than ± 1 .

Finally, for completeness, we mention some interesting connections with the case where the exponent group U does not contain -1 . Orientably regular maps with this property are known as *chiral*. In [4], it was shown by a direct permutation construction that for every pair (d, m) such that $1/d + 1/m \leq \frac{1}{2}$, there exist infinitely many finite orientably regular but chiral maps of type (d, m) . The same thing was proved in [7] by a different method, with the help of holomorphic differentials.

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